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Adiabatic phases and group theory

J Wudka

University of California at Riverside, Department of Physics, Riverside, CA 92521 0413, USA†

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Abstract. The construction of the adiabatic connection is studied in the case where the symmetry of a Hamiltonian is broken explicitly by a slowly varying perturbation. The type of time variation of the perturbation corresponds to the one generated by the symmetry group of the unperturbed Hamiltonian. It is proven that the adiabatic connection for this type of system is completely determined by the group structure, up to a set of reduced matrix elements: systems with the same symmetries will have adiabatic connections differing at most in these reduced matrix elements. Several examples are detailed.

1. Introduction

The description of the adiabatic approximation, as improved by Berry's formalism [1], provides a framework which relates a variety of subjects ranging from field-theory anomalies [2] to the dynamics of an atom in a slowly varying electromagnetic field [1]. In the present paper this formalism will be applied to a class of Hamiltonians which are of interest both phenomenologically and because of their mathematical properties. Such Hamiltonians consist of a time independent term H_0, invariant under a semi-simple compact Lie group $\mathcal{G}$, together with a slowly varying perturbation, H' which is invariant under a subgroup $\mathcal{H} \subset \mathcal{G}$. The time variation of H' is assumed to be sufficiently slow for the adiabatic approximation to be valid.

This class of Hamiltonians is characteristic of systems interacting with an external field. In many such cases H' is a function of a set of irreducible tensor operators under $\mathcal{G}$ [3]; we shall not, however, assume any specific form for H' and rely solely on the above mentioned group properties for the subsequent calculations.

The time dependence in H' will be assumed to be generated by a (sufficiently slow) $\mathcal{G}$ rotation [4]; this allows for a compact expression for the adiabatic connection. Moreover, as will be seen below, the adiabatic connection is uniquely determined up to a set of reduced matrix elements; so that models with the same group and degeneracies will have connections differing at most in these reduced matrix elements. This fact was observed in [3] for the case of $\mathcal{G} = SU(2)$, $\mathcal{H} = U(1)$ using topological arguments. As will be proven, this is a completely general feature which can be obtained using elementary group theory.

Though several of the results derived below have appeared previously in the literature, the method of calculation here described, together with the general expression for the adiabatic connection are, as far as the author is aware, new.

† Email address: WUDKA@UCRPHYS

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This paper is organized as follows. In the next section the group structure is developed to the point where an expression for the adiabatic connection amenable for calculations is obtained. Section 3 briefly describes some interesting types of degeneracies. Several examples are worked out in section 4, while comments and conclusions appear in the last section.

2. Hamiltonian and adiabatic connection

As mentioned above the Hamiltonians under consideration are of the form $H_0 + H'$. H_0 is time independent and invariant under a compact semi-simple Lie group $\mathcal{G}$, while H' is time dependent and invariant under a subgroup $\mathcal{H} \subset \mathcal{G}$. Below we will briefly describe the properties associated with this group structure; in the following $D^{(\ell)}$ will always denote the ℓth irreducible representation of $\mathcal{G}$.

Let $\mathbf{g}$ be the algebra of $\mathcal{G}$ and $\mathbf{h}$ that of $\mathcal{H}$; since $\mathcal{G}$ is compact, the corresponding structure constants are completely anti-symmetric and the Cartan metric can be taken to be proportional to the identity [5]. The basis for $\mathbf{h}$ will be denoted by $\{Q_r\}$, and the basis for $\mathbf{g} - \mathbf{h}$ by $\{Q'_I\}$; small case Latin indices from the end of the alphabet, r, s, etc. will denote the generators for $\mathbf{h}$; capital Latin indices from the middle of the alphabet, I, J, etc. will denote the generators of $\mathbf{g} - \mathbf{h}$. It follows that

$$[Q_r, Q'_I] = -\lambda^J_{rI} Q'_J \tag{2.1}$$

where no term proportional to Q_r appears on the right-hand side due to the anti-symmetry of the structure constants, and because the Q_r close into and algebra. From the Jacobi identity it follows that the matrices λ_r carry a representation (reducible in general) of $\mathbf{h}$. Therefore the Q'_I transform as (reducible) tensor operators under $\mathcal{H}$. Since $\mathcal{G}$ is semi-simple, it follows that the representation carried by the λ_r is fully reducible [5]; the basis $\{Q'_I\}$ can then be replaced by sets of irreducible tensor operators under $\mathcal{G}$ [7]; this fact will be used below.

Requiring $\mathcal{G}$ to be compact allows the decomposition $\mathbf{h} = \mathbf{s} \oplus \mathbf{n}$, where $\mathbf{s}$ is a semi-simple algebra, and $\mathbf{n}$ is Abelian [7]. Let $\mathcal{S}$ and $\mathcal{N}$ be the groups generated by $\mathbf{s}$ and $\mathbf{n}$ respectively; the irreducible representations of $\mathcal{S}$ will be labelled by the letters R, S,etc. and those of $\mathcal{N}$ by ρ, σ,etc.

Denote by T_u the anti-Hermitian generators of the unitary representation of $\mathcal{G}$ in the Hilbert space of the system, so that a group element g corresponds to an operator $U(g) = \exp(T_u \theta^u)$. Using this and the above properties of $\mathcal{G}$ implies that the generators T_u can be decomposed into

$$\begin{aligned} &\{\tilde{Q}_i\}: \quad \text{basis of } \mathbf{s} \\ &\{N_a\}: \quad \text{basis of } \mathbf{n} \\ &\{Q_k^{(R;\rho)}\}: \quad \text{basis of } \mathbf{g} - \mathbf{h} \end{aligned} \tag{2.2}$$

where, for simplicity, the symbols $\tilde{Q}_i$, N_a, etc. are used to represent both the abstract Lie algebra elements and their realization as operators. The previous basis of $\mathbf{g}-\mathbf{h}$ has been replaced, as mentioned before, by the set $\{Q_k^{(R;\rho)}\}$, which are irreducible tensor operators under $\mathcal{S}$ and $\mathcal{N}$ (k labels the element of the Rth irreducible representation

under S) :

$$[\tilde{Q}_i, Q_k^{(R;\rho)}] = \Delta^{(R)}(\tilde{Q}_i)_k^l Q_l^{(R;\rho)}$$
$$[N_a, Q_k^{(R;\rho)}] = \mathrm{i}\rho_a Q_k^{(R,\rho)} \tag{2.3}$$

where $\Delta^{(R)}$ denotes the Rth irreducible representation of S, and the ρ_a are real numbers. As previously mentioned, the irreducible representations of $\mathcal{S}$ will be labelled by the letters R, S, etc. and those of $\mathcal{N}$ by ρ, σ,etc; the letters l, k in the above equation denote the indices of the matrices $\Delta^{(R)}$, and they also label the elements of the $(R;\rho)$ irreducible representation; the summation convention is implied.

As mentioned in the introduction, the type of perturbations $H'(t)$ which will be considered are of the form [4]

$$H'(t) = U(g)H'(0)U(g)^\dagger \qquad g = g(t) \quad g(0) = e \tag{2.4}$$

where $g(t)$ corresponds to a given curve in $\mathcal{G}$, and e is the identity in $\mathcal{G}$. The eigenstates of the Hamiltonian for $t = 0$, denoted by $|s\rangle$, can be classified according to their transformation properties under $\mathcal{S}$ and $\mathcal{N}$,

$$|s\rangle = |R, m; n; \zeta\rangle \tag{2.5}$$

where R denotes the irreducible representation of $\mathcal{S}$, m the state within this irreducible representation; n denotes a set $\{n_a\}$, the eigenvalues of the N_a (the generators of **n**); and ζ summarizes the remaining quantum numbers required to uniquely specify $|s\rangle$.

The states of the time dependent Hamiltonian are [4] $U(g)|s\rangle$, $g = g(t)$, so that the adiabatic connection, denoted by $A^{(\mathcal{G})}$, is given by

$$A_{s's}^{(\mathcal{G})} = \mathrm{i}\langle s'|\{U(g)^\dagger \mathrm{d}U(g)\}|s\rangle \tag{2.6}$$

with the restriction that the states $|s\rangle$ and $|s'\rangle$ are degenerate. This expression for $A^{(\mathcal{G})}$ can be simplified by noting that

$$U(g)^\dagger \mathrm{d}U(g) = \Theta_v^w \mathrm{d}\theta^v T_w \qquad \Theta = \left[\frac{\exp(\theta t) - 1}{\theta t}\right] \tag{2.7}$$

where the matrices t generate the adjoint representation of $\mathcal{G}$. The matrix Θ can be expressed in terms of any (unitary) irreducible representation of $\mathcal{G}$ by the relation

$$\mathrm{d}\theta^w \Theta_w^v = \frac{1}{\kappa_\ell} \mathrm{tr}\{D^{(\ell)}(T_v)^\dagger D^{(\ell)}(g)^\dagger \mathrm{d}D^{(\ell)}(g)\} \tag{2.8}$$

where the normalization constant κ_ℓ is determined by

$$\mathrm{tr}\{D^{(\ell)}(T_v)^\dagger D^{(\ell)}(T_u)\} = \kappa_\ell \delta_v^u$$

Thus the evaluation of $A^{(\mathcal{G})}$ requires the matrix elements of the generators T_u. These are trivial to obtain for the elements of **s** and **n**; for the elements of **g** − **h**, such

matrix elements involve the Clebsch–Gordan coefficients of $\mathcal{H}$ to which we now turn. Following Cornwell and van den Broek [6] these coefficients are specified as follows: the product of two irreducible representations of a group, $S \otimes S'$, will contain a given irreducible representation R a number $\nu^R_{S,S'}$ times, so that the product is expressed as

$$\chi^{(R,\alpha)}_k = \sum_{m,m'} \left(\begin{matrix} S & S' \\ m & m' \end{matrix} \middle| \begin{matrix} R & \alpha \\ k & \end{matrix} \right) \varphi^{(S)}_m \psi^{(S')}_{m'} \qquad 0 \leqslant \alpha \leqslant \nu^R_{S,S'} \tag{2.9}$$

where χ, φ and ψ denote the basis vectors of the corresponding irreducible representations; and the complex numbers

$$\left(\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix} \middle| \begin{matrix} \cdot & \cdot \\ \cdot & \end{matrix} \right)$$

are the Clebsch–Gordan coefficients.

With these preliminaries the required matrix elements are given by

$$\begin{aligned} &\langle S', m'; n'; \zeta' | \, \tilde{Q}_i \, | S, m; n; \zeta \rangle = \Delta^{(S)}_{m'm}(\tilde{Q}_i) \delta_{S'S} \delta_{n'n} \delta_{\zeta'\zeta} \\ &\langle S', m'; n'; \zeta' | \, N_a \, | S, m; n; \zeta \rangle = i n_a \delta_{S'S} \delta_{n'n} \delta_{\zeta'\zeta} \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\langle S', m'; n'; \zeta' | \, Q^{(R;\rho)}_k \, | S, m; n; \zeta \rangle \\ &\quad = \delta_{n+\rho,n'} \sum_{\alpha=1}^{\nu^R_{S,S'}} \left(\begin{matrix} S & R \\ m & k \end{matrix} \middle| \begin{matrix} S' & \alpha \\ m' & \end{matrix} \right) \mathcal{Q}(SS'|R\alpha|nn'|\zeta\zeta') \end{aligned}$$

where $\Delta^{(S)}$ denotes the Sth irreducible representation of $\mathcal{S}$, and $\mathcal{Q}$ denotes a reduced matrix element and, $\delta_{n+\rho,n'}$ vanishes unless $n_a + \rho_a = n'_a$ for all a. The (generalized) Wigner–Eckart theorem [6] was used in obtaining the last expression .

Substituting these results in (2.6) gives

$$\begin{aligned} A^{(\mathcal{G})}_{s's} = \mathrm{i} \, \mathrm{d}\theta^w \Bigg[&\sum_i \Theta^i_w \Delta^{(S)}_{m'm}(\tilde{Q}_i) \delta_{nn'} \delta_{SS'} \\ &+ \mathrm{i} \sum_a \Theta^a_w n_a \delta_{m'm} \delta_{SS'} \delta_{nn'} + \sum_{R,k;\rho} \delta_{n+\rho,n'} \Theta^{([R,k];\rho)}_w \\ &\times \sum_{\alpha=1}^{\nu^R_{S,S'}} \left(\begin{matrix} S & R \\ m & k \end{matrix} \middle| \begin{matrix} S' & \alpha \\ m' & \end{matrix} \right) \mathcal{Q}(SS'|R\alpha|nn'|\zeta\zeta') \Bigg]. \end{aligned} \tag{2.11}$$

The range of the index v in (2.8) has been separated according to the classification (2.2); the labels $(S, m; n; \zeta)$ refer to the state $|s\rangle$, while their primed counterparts refer to $|s'\rangle$; the rest of the notation is self evident.

Equation (2.11) is the main result of this paper, the following considerations will deal with simplifications of the above expressions under various assumptions. The contributions to $A^{(\mathcal{G})}$ which appear only when $S' = S$ and $n'_a = n_a$ will be labelled 'diagonal'; the remaining terms will be called 'off-diagonal'.

Models having the same group structure (and degeneracies) will have adiabatic connections differing only in the reduced matrix elements, so that we can talk about families of models whose adiabatic connections are parametrized by the Q. The terms Clebsch–Gordan coefficients will vanish in case the irreducible representation S' is not contained in $S \otimes R$, or if, for every ρ, $\rho_a \neq n'_a - n_a$ for some a; in general, however, these conditions are not satisfied.

It is also worth pointing out that the same expression will be valid for non-compact groups in those special cases for which the generators of $\mathbf{g} - \mathbf{h}$ are tensor operators under $\mathbf{h}$. It must also be noted that for non-compact groups the expression Θ in (2.8) must involve the Cartan metric which in this case is not proportional to the identity. An example where these remarks are relevant is the case where $\mathcal{G} = \mathrm{SO}(3,1)$ and $\mathcal{H} = \mathrm{SO}(3)$, which will be studied below.

The expression (2.11) leads to simple results only in the special cases, the difficulty being both in the explicit evaluation of the reduced matrix elements Q and of the tensor Θ. The explicit expression for Θ is available only when $\mathcal{G}$ is a product of several $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ factors. If we then consider only these cases, then the determination of the reduced matrix elements is straightforward: $\mathcal{H}$ will also be a product of $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ factors and, therefore, the representation matrices Δ are known explicitly; such a situation will be studied below. In contrast, when $\mathcal{G}$ is a more complicated group (excluding the non-compact partners of $\mathrm{SU}(2)$), no explicit expressions for $A^{(\mathcal{G})}$ are available.

To explicitly determine the reduced matrix elements one must first find the states (2.5), and then use (2.10) to determine Q. The advantage of the procedure is the usual one: one needs to evaluate the matrix elements for one state in a given irreducible representation, the Clebsch–Gordan coefficients determine all the other matrix elements. The Clebsch–Gordan coefficients can be obtained using standard methods [6].

3. Degeneracies

The construction of the adiabatic connection requires the selection of all states corresponding to a given energy. While in many instances he structure of the degeneracies can be obtained by symmetry considerations, this is not always the case, and such 'unexplained' degeneracies are commonly labelled 'accidental'. This terminology is very ambiguous; for example, the special degeneracies present in the Coulomb problem appear because the actual symmetry group is $\mathrm{SO}(3,1)$ [5], a fact not immediately apparent in the coordinate representation.

In order to avoid these ambiguities, a precise (if somewhat ad hoc) definition of accidental degeneracy for the systems under consideration will be given as follows: a degeneracy will be called accidental if it is not due to the invariance under $\mathcal{H}$ or due to an invariance under an outer automorphism of $\mathcal{H}$†. For a detailed examination of accidental degeneracies see [8]. In this section degeneracies which occur as a result of invariance under the outer automorphisms of $\mathcal{H}$ will be described.

Consider first the diagonal elements of $A^{(\mathcal{G})}$. These will include, aside from the first two terms in (2.11), those where the representations R and ρ are singlets; in this

† The second case is included so as to encompass symmetries such as time reversal; see section 4.1.

case the Clebsch–Gordan coefficients are equal to one and $\nu^R_{S,S} = 1$. This gives

$$A^{(\mathcal{G};\ \text{diag})}_{s's} = \mathrm{i}\mathrm{d}\theta^w \left[\sum_i \Theta^i_w \Delta^{(S)}_{m'm}(\tilde{Q}_i) + i \sum_a \Theta^a_w n_a \delta_{m'm} + \sum_{(R;\rho)\ =\text{singlet}} \Theta^{(R;\rho)}_w Q(S|R|n|\zeta\zeta')\delta_{m',m} \right] \delta_{n'n}\delta_{S'S} \tag{3.1}$$

where the notation indicates that only those terms corresponding to R equal to a singlet under $\mathcal{S}$ and ρ singlet under $\mathcal{N}$ (i.e. $\rho_a = 0$) should be included. There is no sum over k since the singlet representations are one dimensional. If there are no degeneracies aside from the one in m, the above expression is the complete result.

The off-diagonal entries are

$$A^{(\mathcal{G};\ \text{non-diag})}_{s's} = \mathrm{i}\mathrm{d}\theta^w \left[\sum_{([R,k];\rho)} \delta_{n+\rho,n'} \Theta^{([R,k];\rho)}_w \times \sum_{\alpha=1}^{\nu^R_{S,S'}} \begin{pmatrix} S & R & | & S' & \alpha \\ m & k & | & m' & \end{pmatrix} Q(SS'|R\alpha|nn'|\zeta\zeta') \right] \tag{3.2}$$

with the restriction that $S' \neq S$ or $n' \neq n$, i.e. that $(R;\rho)$ are not singlets under $\mathcal{H}$. To determine whether these contributions are present is complicated by the possible occurrence of accidental degeneracies (as defined above). For simplicity it will be assumed that these do not occur; the only off-diagonal entries will then be a consequence of the invariance under outer automorphisms of the group $\mathcal{H}$.

In view of this last assumption the quantum number(s) ζ are irrelevant; henceforth such labels will not be displayed.

Thus, with the above restrictions in mind, we can state that for any two degenerate states $|s'\rangle$ and $|s\rangle$ such that they do not belong to the same irreducible representation of s and n, there will be an operator P, which commutes with the Hamiltonian H, generates an outer automorphisms of $\mathcal{H}$, and such that $P|s\rangle = |s'\rangle$. The automorphisms can be separated according whether they act on a simple factor subgroup of $\mathcal{H}$ (class 1 automorphisms), or do not (class 2 automorphisms). Class 2 comprises both the automorphisms of $\mathcal{N}$ and those that correspond to the permutation of identical factor groups (whenever there are more than two identical factors in the decomposition of $\mathcal{H}$). A given system can be invariant under both classes of automorphisms simultaneously.

3.1. Class 1

For automorphisms of a simple factor group of $\mathcal{S}$ the Hermitian operator P is determined by the action of the automorphisms on the irreducible representation of the factor group under question carried by the state $|s\rangle$. This, in its turn, is determined by the effect of the automorphism on the corresponding Dynkin diagram, thus [5], $P^2 = 1$ for E_6, $\mathrm{SO}(2n)$, $n \geqslant 5$ and $\mathrm{SU}(n+1)$, $n \geqslant 2$. For all other cases $P = 1$, except for $\mathrm{SO}(8)$, where $P^3 = 1$.

The automorphisms of $\mathrm{SU}(n \geqslant 3)$ and $\mathrm{SO}(4n+2)$, $n \geqslant 2$ correspond to complex conjugation of the generators; therefore if, for example, $\mathrm{s} = \mathrm{SU}(n)$ with $n \geqslant 3$,

then degeneracies will occur provided $R \subset S \otimes S$. For $SO(4n)$, $n \geqslant 3$, the automorphism corresponds to the exchange of two real fundamental (spinor) representations. Finally, $SO(8)$ automorphisms have, aside from the previous exchange of the two spinor representations, also an automorphisms which interchanges the defining and a spinor representations.

Given a Hamiltonian invariant under this type of transformation the determination of (3.2) is straightforward from group theory: the group automorphisms determine the degenerate irreducible representations, while the Clebsch–Gordan coefficients are determined using, for example, the method of [6]. An example of this procedure is presented in section 4.2 below.

3.2. Class 2

Assume now that P is an automorphism of $\mathcal{N}$. For $U(1)$ the only outer automorphism is equivalent to complex conjugation (i.e. to the change in sign of the generator). Therefore the action of P is given by $N_a \to \epsilon_a N_a$, $\epsilon_a = \pm 1$. This type of degeneracies include those resulting from invariance under space, time or charge inversion in many physically interesting systems; the degenerate states correspond to $\{n_a\}$ and $\{n'_a = \epsilon_a n_a\}$ (no sum over a).

There is however the possibility that H is invariant under a permutation of the N_a, denoted by $N_a \to N_{P(a)}$. If this is the case then the degenerate states correspond $\{n_a\}$ and $\{n'_a = \epsilon_a n_{P(a)}\}$. This case will be considered below in section 4.1.

The last type of automorphism occurs when the group S has identical factor groups. In this case P corresponds to the exchange of the corresponding groups and the action on the states corresponds to the exchange of the corresponding irreducible representations. States thus related, however, do not contribute to (3.2), as is proven next.

Suppose that $\to$ has two identical simple factors with basis $\{\tilde{Q}_i\}$ and $\{\tilde{Q}_I\}$ satisfying $[\tilde{Q}_i, \tilde{Q}_I] = 0$. Just as in (2.1) the Q'_I transform as tensor operators under both these sets; let λ and $\bar{\lambda}$ be the corresponding matrices. Consider then the Jacobi identity

$$0 = [\tilde{Q}_I, [\tilde{Q}_i, Q'_I]] + [\tilde{Q}_i, [Q'_I, \tilde{Q}_I]] + [Q'_I, [\tilde{Q}_I, \tilde{Q}_i]] = [\lambda_i, \bar{\lambda}_I]_{IJ} Q'_J \tag{3.3}$$

which implies that the λ and $\bar{\lambda}$ commute. It follows that if a subset of the Q'_I carry a non-trivial representation for one factor, they carry a trivial representation for the other factor. Thus, when reduced to their irreducible transforming sets, the Q' take the form $G \otimes 1$ or $1 \otimes \bar{G}$; where G and $\bar{G}$ are irreducible tensor operators under the corresponding factors; since $\tilde{Q}_i$ and $\tilde{Q}_I$ are also of this form, we conclude that all generators of the group can be written thus. Now consider two states, $|s, m; \bar{s}, \bar{m}\rangle$ and $P|s, m; \bar{s}, \bar{m}\rangle = |\bar{s}, \bar{m}; s, m\rangle$; where s and $\bar{s}$ label some irreducible representations (and m, $\bar{m}$ the states in them) and P is the permutation operator exchanging the factors in the group (other labels are suppressed for brevity); it is assumed that $s \neq \bar{s}$. It follows that the matrix element of any generator contains a factor $\langle \bar{s}, \bar{m} | s, m\rangle$ (or its conjugate) which is zero. Therefore the corresponding contribution to (3.2) vanishes as claimed.

4. Examples

The method described above for obtaining the adiabatic connection consists in first determining $\mathcal{S}$ and $\mathcal{N}$ together with the representations $(R; \rho)$ present in the Q'_I.

Then evaluating Θ using (2.8). Next the energy degeneracies should be found; and, finally, the non-vanishing Clebsch–Gordan coefficients must obtained. The results of these steps produce $A^{(\mathcal{G})}$. This procedure will be applied to some illustrative examples.

4.1. Example 1

Consider first the case where $\mathcal{H} = \mathcal{N}$. Then the connection is given by

$$A^{(\mathcal{G})}_{s's} = \mathrm{i}\mathrm{d}\theta^v \left[i \sum_a \Theta^a_v n_a \delta_{nn'} + \sum_\rho \delta_{n+\rho,n'} \Theta^\rho_v Q(nn') \right] \tag{4.1}$$

it being understood that n and n' correspond to degenerate states.

This situation is realized when H' is a function of the Cartan generators of $\mathcal{G}$ [4, 9]. For simplicity it will be assumed that all the Cartan generators are present in H', and that there are no accidental degeneracies. In this case the $\{Q^{(\rho)}\}$ are the roots of $\mathbf{g}$, so that the numbers ρ_a correspond to the elements of the root associated with $Q^{(\rho)}$ (there is no label R since there is no $\mathcal{S}$). Moreover H will also commute with the Casimir operators for $\mathcal{G}$ and the states can be labelled by their weight and Casimir eigenvalues. A simple calculation shows that in this case [5]

$$Q(n, n+\rho) = \mathrm{i}\sqrt{p\,(q+1)}\sqrt{\rho_a \rho_a / 2} \tag{4.2}$$

where the integers p and q depend on n and ρ, and are determined by the condition $Q^{(\rho)}|n + p\rho\rangle = 0$ and $Q^{(-\rho)}|n - q\rho\rangle = 0$ (there are no S, m labels since $\mathcal{H} = \mathcal{N}$). The conventions used are such that $Q^{(\rho)\dagger} = -Q^{(-\rho)}$, so that $(\Theta^\rho_v)^* = \Theta^{-\rho}_v$.

As a specific case assume $\mathcal{G} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathcal{H} = \mathrm{U}(1) \times \mathrm{U}(1)$ with H invariant under any permutation of the factor groups. To determine the connection we will use the following conventions: the generators of $\mathrm{SU}(2)$ are denoted by T_u, $u = \pm, 0$, with $[T_\pm, T_0] = \pm \mathrm{i} T_\pm$, so that $\rho = \pm 1$; and the corresponding $\mathrm{U}(1)$ can be assumed to be generated by T_0. Using (2.8) with a spin of one representation of $\mathrm{SU}(2)$ in terms of Euler angles ψ, θ, ϕ (in the convention of [10]) gives

$$\begin{aligned} \mathrm{d}\theta^v \Theta^0_v &= \mathrm{d}\psi + \mathrm{d}\phi \cos\theta = \Gamma \\ \mathrm{d}\theta^v \Theta^+_v &= -\frac{1}{\sqrt{2}} \exp(\mathrm{i}\psi)\,(\mathrm{d}\theta - \mathrm{i}\mathrm{d}\phi \sin\theta) = -\frac{1}{\sqrt{2}} \Upsilon . \end{aligned} \tag{4.3}$$

The normalization used is $\mathrm{tr}\, T^\dagger_u T_v = 2\delta_{u,v}$, $T^\dagger_+ = -T_-$. The states for $\mathrm{SU}(2)$ are labelled by their Casimir eigenvalue j and their weight $|j, n\rangle$, with $|n| \leqslant j$, then $p = j - n$, $q = j + n$.

Subindices '1' and '2' will be used to identify the two $\mathrm{U}(1)$ factors and the corresponding $\mathrm{SU}(2)$ parent groups; the states are denoted by $|j_1, n_1; j_2, n_2\rangle$. Since $T_\pm$ change the corresponding n_i by ± 1, and since the only degeneracies are assumed to be produced by the outer automorphisms of $\mathcal{H}$, it follows that there are three types of subspaces relevant for the evaluation of $A^{(\mathcal{G})}$ (the labels $j_{1,2}$ are fixed and omitted for brevity):

(a) The one dimensional space $\mathcal{W}(n_1, n_2) = \{|n_1, n_2\rangle : |n_i| \neq \frac{1}{2},\ i = 1, 2\}$; the corresponding the expression for the adiabatic connection can be obtained directly from (3.1) yielding

$$A^{(\mathcal{G})}[\mathcal{W}(n_1, n_2)] = -\sum_{i=1,2} n_i \Gamma_i \tag{4.4}$$

(b) The two dimensional spaces $\mathcal{X}_1(n_2) = \{|\pm\tfrac{1}{2}, n_2\rangle : |n_2| \neq \tfrac{1}{2}\}$ and $\mathcal{X}_2(n_1) = \{|n_1, \pm 1/2\rangle : |n_1| \neq 1/2\}$. For $\mathcal{X}_1(n_2)$ the adiabatic connection is, from (4.1) and (4.2)

$$A^{(\mathcal{G})}[\mathcal{X}_1(n_2)] = \begin{pmatrix} -\frac{1}{2}\Gamma_1 - n_2\Gamma_2 & (j_1 + \frac{1}{2})\Upsilon_1 \\ (j_1 + \frac{1}{2})\Upsilon_1^* & \frac{1}{2}\Gamma_1 - n_2\Gamma_2 \end{pmatrix} \tag{4.5}$$

Similarly

$$A^{(\mathcal{G})}[\mathcal{X}_2(n_1)] = \begin{pmatrix} -\frac{1}{2}\Gamma_2 - n_1\Gamma_1 & (j_2 + \frac{1}{2})\Upsilon_2 \\ (j_2 + \frac{1}{2})\Upsilon_2^* & \frac{1}{2}\Gamma_2 - n_1\Gamma_1 \end{pmatrix} \tag{4.6}$$

where Γ_i, Υ_i are given in (4.3).

(c) The four dimensional space $\mathcal{Y} = \{|n_1, n_2\rangle : |n_i| = \tfrac{1}{2},\ i = 1, 2\}$; again from (4.1) and (4.2) we obtain

$$A^{(\mathcal{G})}[\mathcal{Y}] = \begin{pmatrix} -\frac{1}{2}(\Gamma_1 + \Gamma_2) & (j_2 + \frac{1}{2})\Upsilon_2 & (j_1 + \frac{1}{2})\Upsilon_1 & 0 \\ (j_2 + \frac{1}{2})\Upsilon_2^* & -\frac{1}{2}(\Gamma_1 - \Gamma_2) & 0 & (j_1 + \frac{1}{2})\Upsilon_1 \\ (j_1 + \frac{1}{2})\Upsilon_1^* & 0 & \frac{1}{2}(\Gamma_1 - \Gamma_2) & (j_2 + \frac{1}{2})\Upsilon_2 \\ 0 & (j_1 + \frac{1}{2})\Upsilon_1^* & (j_2 + \frac{1}{2})\Upsilon_2^* & \frac{1}{2}(\Gamma_1 + \Gamma_2) \end{pmatrix}$$

where Γ_i, Υ_i are given in (4.3).

The adiabatic connections for $\mathcal{X}_i$ are identical to the one where $\mathcal{G} = \mathrm{SU}(2)$, $\mathcal{H} = \mathrm{U}(1)$, and has been studied in detail in [11, 3]. It is included here to illustrate the use of (2.11) and also to demonstrate that the results of [3] can be obtained directly from the group structure of the problem. A simple realization of this situation corresponding to $\mathcal{W}$ and $\mathcal{X}_1$ with $n_2 = 0$ is the case of an atom in the presence of an external electric or magnetic field, which rotates slowly in time with constant magnitude [11]; $\mathcal{H} = \mathrm{U}(1)$ then corresponds to rotations around the field's axis. If the atom has an even number of electrons, then n_1 is an integer so that, barring accidental degeneracies, the connection is diagonal and equal to $-n_1\Gamma$. However, if the atom has an odd number of electrons then n_1 is a half-odd integer and the connection can acquire off-diagonal elements. Such is the case when Kramers' degeneracy is present in an atom with magnetic quantum number equal to $\frac{1}{2}$. Then the above results for $\mathcal{X}_1(n_2 = 0)$ determine the corresponding connection, which coincides with the expression found by Mead [11].

4.2. Example 2

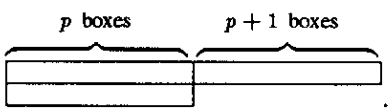
To illustrate the effect of invariance under the outer automorphisms of s, consider the case where $\mathcal{G} = \mathrm{SU}(N)$ and $\mathcal{H} = \mathrm{SU}(M)$, $(M \leqslant N - 1)$. The representations R carried by the Q_I' are $N - M$ copies of the fundamental of $\mathrm{SU}(M)$ (denoted by

M), $N-M$ copies of its complex conjugate (denoted by $\bar{\mathbf{M}}$), and $(N-M)^2$ SU(M) singlets (denoted by **1**). For any representation S of SU(M) the action of P is simply to replace it by its complex conjugate: $PS = \bar{S}$, therefore the off-diagonal elements of $A^{(\mathcal{G})}$ will be non-zero if and only if $S \otimes S$ contains **1**, **M** or $\bar{\mathbf{M}}$. Using a tensor realization of S shows that the last two cases are impossible for $M \neq 3$, and that $S \otimes S$ can contain singlets only if S is real. It then follows that (for $M \neq 3$) (3.2) vanishes while (3.1) gives

$$A^{(\mathcal{G})}_{s's} = \mathrm{i}\mathrm{d}\theta^v \times \left[\sum_i \Theta^i_v \Delta^{(S)}_{m'm}(\tilde{Q}_i) + \sum_{R=1}^{(N-M)^2} \Theta^R_v Q(S|R)\right]\delta_{S'S} \tag{4.7}$$

where the second sum is over the $(N-M)^2$ singlet representations carried by the Q^i_I and vanishes except for real S. The reduced matrix elements are zero unless S is real (in which case they are purely imaginary).

For $M = 3$ ($N \geqslant 4$) the diagonal elements are again given given by (4.7); but now there is an off-diagonal contribution. Let S_p be the SU(3) representation characterized by the following Young tableaux



(4.8)

Then the off-diagonal elements are non-vanishing when $R = \mathbf{3}$, $S = S_p$, $S' = \bar{S}_p$, or when $R = \bar{\mathbf{3}}$, $S = \bar{S}_p$, $S' = S_p$; where **3** denotes the fundamental representation, and an overbar indicates the complex conjugate representation. In this case we have $\nu^{\mathbf{3}}_{S_p,\bar{S}_p} = 1$ and (3.2) becomes

$$\begin{aligned} A^{(\mathcal{G};\ \text{non-diag})}_{s's} &= \sum_k \Phi_k \left(\begin{array}{cc|cc} S_p & \mathbf{3} & \bar{S}_p & 1 \\ m & k & m' & \end{array}\right) Q(S_p \bar{S}_p|\mathbf{3}) \\ \Phi_k &= \mathrm{i}\mathrm{d}\theta^v \Theta^{([\mathbf{3};k])}_v \end{aligned} \tag{4.9}$$

or its complex conjugate. For example, if $S_p = \mathbf{3}$ then

$$\left(\begin{array}{cc|cc} S_p & \bar{\mathbf{3}} & \bar{S}_p & \bar{1} \\ m & k & m' & \end{array}\right) = \epsilon_{mkm'}/\sqrt{2}$$

and

$$A^{(\mathcal{G};\ \text{non-diag})}_{s's} = \sum_k \epsilon_{mkm'}\Phi_k/\sqrt{2} \qquad (S_p = \mathbf{3}) \tag{4.10}$$

The forms Φ_k cannot be explicitly obtained in general, but can be evaluated given any particular trajectory $g(t)$ in (2.4).

4.3. Example 3

As a final example take the case $\mathcal{G} = \mathrm{SO}(3,1)$ and $\mathcal{H} = \mathcal{S} = \mathrm{SO}(3)$; the language of special relativity will be used. It will be assumed that there are no degeneracies other than those corresponding to $\mathcal{H}$, so that only the case $S' = S$ needs to be considered.

The generators of $\mathrm{SO}(3,1)$ can be segregated into boost generators K_i $(i = 1,2,3)$, corresponding to the $Q^{(R)}$, and rotation generators J_i $(i = 1,2,3)$, corresponding to the $\tilde{Q}_i$. The boosts transform as vector operators under rotations, i.e. $R = 1$.

Using the fact that for $\mathrm{SO}(3)$ $\nu^R_{S,S} = 1$, then for $R = 1$, we obtain

$$\left(\begin{array}{cc|cc} S & 1 & S & 1 \\ m & k & m' & \end{array}\right) = \frac{\delta_{m+k,m'}}{\sqrt{S(S+1)}} \begin{cases} \mp\sqrt{(S \pm m')(S \mp m' + 1)/2} & k = \pm 1 \\ m' & k = 0 \end{cases} \tag{4.11}$$

where $S(S+1)$ and $S'(S'+1)$ are the eigenvalues of the $\mathrm{SO}(3)$ Casimir operator for the corresponding irreducible representation, thus S, S' label the representation; they are both integers.

The calculation of Θ using (2.8) is simplified by noting that g is defined up to an $\mathrm{SO}(3)$ transformation, thus we can take

$$D^{(\ell)}(g) = \exp\left\{\sum_i m_i D^{(\ell)}(K_i)\right\} \tag{4.12}$$

Using for $D^{(\ell)}$ the fundamental representation of $\mathrm{SO}(3,1)$ realized as 4×4 matrices in the conventions of [12], together with a polar parametrization of $\mathbf{m}$ (whose components are m_i)

$$\mathbf{m} = \mu(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{4.13}$$

yields, after a straightforward use of (2.11)

$$A^{(\mathcal{G})} = \mathrm{i}\Delta^{(S)}(\omega_i \tilde{Q}_i) \tag{4.14}$$

In this equation $\Delta^{(S)}$ denotes the Sth irreducible representation of $\mathrm{SO}(3)$, and the one forms ω are

$$\begin{aligned} \omega &= \omega_T + \tilde{Q}_S\, \zeta_T \\ \omega_T &= (\cosh\mu - 1)(\hat{\phi}\, \mathrm{d}\theta - \hat{\theta}\, \sin\theta\, \mathrm{d}\phi) \\ \zeta_T &= [\hat{r}\, \mathrm{d}\mu + \sinh\mu\, (\hat{\theta}\, \mathrm{d}\theta + \hat{\phi}\, \sin\theta\, \mathrm{d}\phi)] \end{aligned} \tag{4.15}$$

where $\tilde{Q}_S = Q(S)/\sqrt{S(S+1)}$.

The nomenclature is chosen to indicate the close relation to the quantities found in the calculation of the Thomas precession [12]: ω_T is (minus) the Thomas frequency, while ζ_T is the Thomas boost. More precisely, if θ and ϕ denote the polar angles of the boost velocity v and $\tanh\mu$ its magnitude, then a boost with a velocity $-v$ followed by another boost to velocity $v + \mathrm{d}v$ equals a boost to velocity ζ_T followed by a rotation by an angle ω_T.

This example is relevant for the study of a particle under the action of a slowly varying (in time) time-like electromagnetic potential. It also provides the correct phase when considering the adiabatic rotation of fermionic states [13].

5. Conclusions

It has been shown that elementary group theory is sufficient to completely describe the adiabatic connection for the systems under consideration. The connections obtained are determined by the group structure up to a set of reduced matrix elements which depend on the details of the Hamiltonian. Therefore we can organize these models in families according to the group (and degeneracies). This is summarized by the equation (2.11), this paper's main result.

The examples show that the method proposed here is amenable for practical calculations. Of course, detailed properties of the various representations of the group $\mathcal{G}$ are needed in order to obtain concrete results but, given these, obtaining the adiabatic connection is straightforward.

The last term in expression (2.11) is determined by the representations carried by the Q'_I. These are simple in most cases of physical interest: for example, if $\mathcal{G} = \mathrm{SU}(N)$ and $\mathcal{H} = \mathrm{SU}(M)$, then, as mentioned in section 4, the Q'_I carry the fundamental representation of $\mathrm{SU}(M)$, its complex conjugate, and the singlet representation. Still the author knows of no general description of the representations R appearing in the Q'_I.

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References

[1] Berry M V 1984 *Proc. R. Soc.* A **45** 392
Simon B 1983 *Phys. Rev. Lett.* **51** 2167
Wilczek F and Zee A 1984 *Phys. Rev. Lett.* **52** 2111
Shapere A and Wilczek F 1989 Geometric phases in physics *Advanced Series in Mathematical Physics 5* (Singapore: World Scientific)
Berry M V 1989 Anomalies, Defects and Phases ... *Lectures given at the Ferrara School of Theoretical Physics during June 1989*
[2] Sonoda H 1986 *Nucl. Phys.* B **226** 410
[3] Segert J 1987 *J. Math. Phys.* **28** 2102
[4] Jordan T F *University of Minnesota Report Print 87-639*
Giavarni G and Onofri E 1989 *J. Math. Phys.* **30** 659
Giavarni G *et al* 1988 *CERN Report CERN-TH.5262/88*
Kosiński G P and Szymanowski L 1989 *Int. J. Mod. Phys.* A **4** 1453
[5] Gilmore R 1974 *Lie Groups, Lie Algebras and some of their Applications* (New York: Wiley)
[6] van den Broek P M and Cornwell J F 1980 *Phys. Status Solidi* b **90** 211
[7] Barut A O and Raczka R 1987 *Theory of Group Representations and Applications* 2nd edn (Singapore: World Scientific)
[8] Moshinsky M, Quesne C and Loyola G 1990 *Ann. Phys.* **198** 103
Moshinsky M 1983 *Found. Phys.* **13** 73
Moshinsky M and Patera J 1975 *J. Math. Phys.* **16** 1866
[9] Jackiw R 1989 *Lecture presented at the 25th Anniversary Conference, ICTP, Trieste, Italy*
Sousa-Gerbert P 1989 *Ann. Phys.* **189** 155
[10] Goldstien H 1956 *Classical Mechanics* (Reading, MA: Addison-Wesley)
[11] Mead C A 1987 *Phys. Rev. Lett.* **59** 161
[12] Jackson J D 1975 *Classical Electrodynamics (Second Edition)* (New York: Wiley)
[13] Grundberg J, Hanson T H and Karlhede A 1990 *Phys. Rev.* D **41** 2642